[This question paper contains 8 printed pages.]

22

Your Roll No. 2023

Sr. No. of Question Paper : 4512

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt **two** parts from each question.

1. (a) Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \text{ and } 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$. (6)

P.T.O.

(b) (i) Prove that the limit of the function

$$f(z)=\left(\frac{z}{\bar{z}}\right)^2$$

as z tends to 0 does not exist.

(ii) Show that

$$\lim_{z\to\infty}\frac{4z^2}{(z-1)^2}=4\ .$$ (3+3=6)

(c) Show that the following functions are nowhere differentiable.

(i) $f(z) = z - \bar{z}$,

(ii) $f(z) = e^y \cos x + ie^y \sin x$. (3+3=6)

(d) (i) If a function f(z) is continuous and nonzero at a point z_0, then show that $f(z) \neq 0$ throughout some neighborhood of that point.

(ii) Show that the function $f(z) = (z^2 - 2)e^{-x}e^{-iy}$ is entire. (3+3=6)

2. (a) (i) Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$|\exp \exp (2z + i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}$.

(ii) Find the value of z such that

$$e^z = 1 + \sqrt{3}\,i \qquad (3.5+3=6.5)$$

(b) Show that

(i) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z;

(ii) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i$ $(n = 0 \pm 1, \pm 2, \ldots.)$. (3.5+3=6.5)

(c) Show that

(i) $\log\log(i^2) = 2\log i$ where

$$\log z = \ln r + i\theta \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right).$$

(ii) $\log\log(i^2) \neq 2\log i$ where

$$\log z = \ln r + i\theta \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right).$$

(3.5+3=6.5)

(d) Find all zeros of sinz and cosz. (3.5+3=6.5)

3. (a) State Fundamental theorem of Calculus.

Evaluate the following integrals to test if Fundamental theorem of Calculus holds true or not :

(i) $\int_0^{\pi/2} \exp(t+it)\,dt$

(ii) $\int_0^1 (3t-i)^2\,dt$ (2+2+2=6)

(b) Let y(x) be a real valued function defined piecewise on the interval $0 \le x \le 1$ as

$$y(x) = x^3\sin(\pi/x),\ 0 < x \le 1 \text{ and } y(0) = 0$$

Does this equation $z = x + iy,\ 0 \le x \le 1$ represent

(i) an arc

(ii) A smooth arc. Justify.

Find the points of intersection of this arc with real axis. (2+2+2=6)

(c) For an arbitrary smooth curve C: $z = z(t),\ a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integral depends only on the end points of C.

State if it is independent of the arc under consideration or not?

Also, find its value around any closed contour.

(3+1+2=6)

(d) Without evaluation of the integral, prove that

$\left|\int_c \frac{1}{z^2+1}dz\right| \le \frac{1}{2\sqrt{5}}$ where C is the straight line segment from 2 to 2 +i. Also, state the theorem used.

(4+2=6)

4. (a) Use the method of antiderivative to show that

$\int_c (z-z_0)^{n-1} dz = 0$, n = ±1, ±2,-- where C is any closed contour which does not pass through the point z_0. State the corresponding result used.

(4+2.5=6.5)

(b) Use Cauchy Gourset theorem to evaluate :

(i) $\int_c f(z)dz$, when $f(z) = \frac{1}{z^2+2z+2}$ and C is the unit circle $|z| = 1$ in either direction.

(ii) $\int_c f(z)dz$, when $f(z) = \dfrac{5z+7}{z^2+2z-3}$ and C is the circle $|z-2| = 2$. (3+3.5=6.5)

(c) State and prove Cauchy Integral Formula.

(2+4.5=6.5)

(d) Evaluate the following integrals :

(i) $\int_c \dfrac{\cos z}{z(z^2+8)}dz$, where C is the positive oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

(ii) $\int_c \dfrac{2s^2-s-2}{s-2}ds$, $|z| \neq 3$ at $z = 2$, where C is the circle $|z| = 3$. (3.5+3=6.5)

5. (a) If a series of complex numbers converges then prove that the nth term converges to zero as n tends to infinity. Is the converse true? Justify.

(6.5)

(b) Find the Maclaurin series for the function $f(z) = \sinh z$. (6.5)

(c) If a series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to f(z) at all points interior to some circle $|z - z_0| = R$, then prove that it is the Taylor series for the function f(z) in powers of $z - z_0$. (6.5)

(d) Find the integral of f(z) around the positively oriented circle $|z| = 3$ when $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$.

(6.5)

6. (a) For the given function $f(z) = \left(\dfrac{z}{2z+1}\right)^3$, show any singular point is a pole. Determine the order of each pole and find the corresponding residue. (6)

(b) Find the Laurent Series that represents the function

$f(z) = z^2 \sin \dfrac{1}{z^2}$ in the domain $0 < |z| < \infty$. (6)

(c) Suppose that $z_n = x_n + iy_n$, $(n = 1,2,3,...)$ and $S = X + iY$. Then show that

$$\sum_{n=1}^{\infty} z_n = S \text{ iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

(6)

(d) If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then show that

$$\int_C f(z)\,dz = 2\pi i\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

(6)

[This question paper contains 8 printed pages.]

23

Your Roll No. 2023

Sr. No. of Question Paper : 4749

Unique Paper Code : 32357614

Name of the Paper : DSE-3 MATHEMATICAL FINANCE

Name of the Course : **B.Sc. (H) Mathematics CBCS (LOCF)**

Semester : VI

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions are compulsory and carry equal marks.

4. Use of Scientific calculator, Basic calculator and Normal distribution tables all are allowed.

1. (a) Explain Duration of a zero-coupon bond. A 5-year bond with a yield of 12% (continuously compounded) pays a 10% coupon at the end of each year.

(i) What is the bond's price?

(ii) Use duration to calculate the effect on the bond's price of a 0.1% decrease in its yield? (You can use the exponential values: e^x = 0.8869, 0.7866, 0.6977, 0.6188, and 0.5488 for x = –0.12, –0.24, –0.36, –0.48, and –0.60, respectively)

(b) Portfolio A consists of 1-year zero coupon with a face value of ₹2000 and a 10-year zero coupon bond with face value of ₹6000. Portfolio B consists of a 5.95-year zero coupon bond with face value of ₹5000. The current yield on all bonds is 10% per annum.

(i) Show that both portfolios have the same duration.

(ii) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

(You can use the exponential values: e^x = 0.905, 0.368, 0.552, 0.861, 0.223 and 0.409 for x = –0.1, –1.0, –0.595, –0.15, –1.5 and –0.893 respectively)

(c) Explain difference between Continuous Compounding and Monthly Compounding. What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

(d) (i) "When the zero curve is upward sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward sloping the reverse is true." Explain.

(ii) Why does loan in the repo market involve very little credit risk?

2. (a) Explain Hedging. How is the risk managed when Hedging is done using?

(i) Forward Contracts; (ii) Options

(b) (i) Suppose that a March call option to buy a share for ₹50 costs ₹2.50 and is held until March. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised?

(ii) It is May, and a trader writes a September put option with a strike price of ₹20. The stock price is ₹18, and the option price is ₹2. Describe the trader's cash flows if the option is held until September and the stock price is ₹25 at that time.

(c) Write a short note on European put options. Explain the payoffs in different types of put option positions with the help of diagrams.

(d) (i) A trader writes an October put option with a strike price of ₹35. The price of the option is ₹6. Under what circumstances does the trader make a gain.

(ii) A company knows that it is due to receive a certain amount of a foreign currency in 6 months. What type of option contract is appropriate for hedging?

3. (a) Draw the diagrams illustrating the effect of changes in volatility and risk-free interest rate on both European call and put option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

(b) Derive the put-call parity for European options on a non-dividend-paying stock. Use put-call parity to derive the relationship between the vega of a European call and the vega of a European put on a non-dividend-paying stock.

(c) A European call option and put option on a stock both have a strike price of ₹20 and an expiration date in 3 months. Both sell for ₹3. The risk-free interest rate is 10% per annum, the current stock price is ₹19, and a ₹1 dividend is expected in 1 month. Identify the arbitrage opportunity open to a trader? ($e^{-0.0083} = 0.9917$)

(d) Find lower bound and upper bound for the price of a 1-month European put option on a non-dividend-paying stock when the stock price is ₹30, the strike price is ₹34, and the risk-free interest rate is 6% per annum? Justify your answer with no arbitrage arguments, ($e^{-0.005} = 0.9950$)

4. (a) Consider the standard one-period model where the stock price goes from S_0 to S_0u or S_0d with $d < 1 < u$, and consider an option which pays f_u or f_d in each case, and assume that the interest rate is r and time to maturity is T. Derive the formula for the no-arbitrage price of the option.

(b) A stock price is currently ₹40. It is known that at the end of one month it will be either ₹42 or ?38. The risk-free interest rate is 6% per annum with continuous compounding. Consider a portfolio consisting of one short call and Δ shares of the stock. What is the value of Δ which makes the portfolio riskless? Using no-arbitrage arguments, find the price of a one-month European call option with a strike price of ₹39? (You can use exponential value: $e^{0005} = 1.005$)

(c) Construct a two-period binomial tree for stock and European call option with

$S_0 = ₹100, u = 1.3, d = 0.8, r = 0.05, T = 1$ year, $K = ₹95$

and each period being of length $\Delta t = 0.5$ year. Find the price of the European call. If the call was American, will it be optimal to exercise the option early? Justify your answer. ($e^{-0.025} = 0.9753$)

(d) What do you mean by the volatility of a stock? How can we estimate volatility from historical prices of the stock?

5. (a) Let S_0 denote the current stock price, σ the volatility of the stock, r be the risk-free interest rate and T denote a future time. In the Black-Scholes model, the stock price S_T at time T in the risk-neutral world satisfies

$$\ln S_T \sim \phi\left[\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where $\phi(m, v)$ denotes a norma distribution with mean m and variance v.

Using risk-neutral valuation, derive the Black-Scholes formula for the price of a European call option on the underlying stock S, strike price K and maturity T.

(b) A stock price follows log normal distribution with an expected return of 16% and a volatility of 35%. The current price is ₹38. What is the probability that a European call option on the stock with an exercise price of ₹40 and a maturity date in six months will be exercised? (You can use values: $\ln(38) = 3.638$, $\ln(40) = 3.689$)

(c) What is the price of a European call option on a non-dividend-paying stock when the stock price is ₹69, the strike price is ₹70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

(You can use exponential values: $e^{-0.0144} = 0.9857$, $\acute{e}^{-0.025} = 0.9753$)

(d) A stock price is currently ₹40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a 2-year period?

6. (a) Discuss theta of a portfolio of options and calculate the theta of a European call option on a non-dividend-paying stock where the stock price is ₹49, the strike price is ₹50, the risk-free interest rate is 5% per annum and the time to maturity is 20 weeks, and the stock price volatility is 30% per annum. ($\ln(49/50) = -0.0202$)

(b) (i) Explain stop-loss hedging scheme.

(ii) What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

(c) Find the payoff from a butterfly spread created using call options. Also draw the profit diagram corresponding to this trading strategy.

(d) Companies X and Y have been offered the following rates per annum on a ₹5 million 10-year investment :

	Fixed rate	Floating rate
Company X	8.0%	LIBOR
Company Y	8.8%	LIBOR

Company X requires a fixed-rate investment; Company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and that will appear equally attractive to X and Y.

[This question paper contains 4 printed pages.]

(24) **Your Roll No** 2023

Sr. No. of Question Paper : 4792

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra – II

Name of the Course : **B.Sc. (H) Mathematics (CBCS – LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** the questions are compulsory.

3. Attempt any **two** parts from each question.

4. Marks of each part are indicated

1. (a) (i) Prove that If F is a field, then F[x] is a Principal Ideal Domain.

   (ii) Is $\mathbb{Z}[x]$, a Principal Ideal Domain? Justify your answer.

   (b) Prove that $<x^2 + 1>$ is not a maximal ideal in $\mathbb{Z}[x]$.

(c) State and prove the reducibility test for polynomials of degree 2 or 3. Does it fail in higher order polynomials? Justify. (4+2,6,6)

2. (a) (i) State and prove Gauss's Lemma.

(ii) Is every irreducible polynomial over $\mathbb{Z}$ primitive? Justify.

(b) Construct a field of order 25.

(c) In $\mathbb{Z}\left[\sqrt{(-5)}\right]$, prove that $1+3\sqrt{(-5)}$ is irreducible but not prime. (4+2.5,6.5,6.5)

3. (a) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows :

$$f_1(x, y, z) = x - 2y,$$
$$f_2(x, y, z) = x + y + z,$$
$$f_3(x, y, z) = y - 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^* and then find a basis for V for which it is the dual basis.

(b) Test the linear operator $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f(0) + f(1)(x + x^2)$ for diagonalizability and if diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. Find an expression for A^n where n is an arbitrary natural number.

(6,6,6)

4. (a) For a linear operator $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T(a, b, c) = (-b + c, a + c, 3c)$, determine the T-cyclic subspace W of $\mathbb{R}^3$ generated by $e_1 = (1, 0, 0)$. Also find the characteristic polynomial of the operator T_W.

(b) State Cayley-Hamilton theorem and verify it for the linear operator $T: P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$, $T(f(x)) = f'(x)$.

(c) Show that the vector space $\mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$ where $W_1 = \{(a, b, 0, 0): a, b \in \mathbb{R})$, $W_2 = \{(0,0, c, 0): c \in \mathbb{R}\}$ and $W_3 = \{(0,0,0, d): d \in \mathbb{R}\}$.

(6.5,6.5,6.5)

5. (a) Consider the vector space $\mathbb{C}$ over $\mathbb{R}$ with an inner product $\langle .,. \rangle$. Let $\bar{z}$ denote the conjugate of z. Show that $\langle .,. \rangle'$ defined by $\langle z, w\rangle' = \langle \bar{z}, \bar{w} \rangle$ for all $z, w \in \mathbb{C}$ is also an inner product on $\mathbb{C}$. Is $\langle .,. \rangle''$ defined by $\langle z, w\rangle'' = \langle z + \bar{z}, w + \bar{w} \rangle$ for all $z, w \in \mathbb{C}$ an inner product on $\mathbb{C}$? Justify your answer.

(b) Let $V = P(\mathbb{R})$ with the inner product $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(t)q(t)\,dt \quad \forall p(x), q(x) \in V$. Compute the orthogonal projection of the vector $p(x) = x^{2k-1}$ on $P_2(\mathbb{R})$, where $k \in \mathbb{N}$.

(c) (i) For the inner product space $V = P_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$ and the linear operator T on V defined by $T(f) = f' + 3f$, compute $T^*(4 - 2t)$.

(ii) For the standard inner product space $V = \mathbb{R}^3$ and a linear transformation $g: V \to \mathbb{R}$ given by $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$, find a vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$.

(6,6,2+4)

6. (a) Prove that a normal operator T on a finite-dimensional complex inner product space V yields an orthonormal basis for V consisting of eigenvectors of T. Justify the validity of the conclusion of this result if V is a finite-dimensional real inner product space.

(b) Let $V = M_{2\times2}(\mathbb{R})$ and $T: V \to V$ be a linear operator given by $T(A) = A^T$. Determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(c) For the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$.

(6.5,6.5,6.5)

[This question paper contains 4 printed pages.]

25 **Your Roll No....** 2023

E

Sr. No. of Question Paper : 4873

Unique Paper Code : 32357610

Name of the Paper : DSE-4 (Number Theory)

Name of the Course : **CBCS (LOCF) – B.Sc. (H) (Mathematics)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts of each question.

4. Question Nos. **1** to **3**, each part carries **6.5** marks and Question Nos. **4** to **6**, each part carries **6** marks.

1. (a) Determine all solutions in the positive integers of the Diophantine equation

$$18x + 5y = 48.$$

(b) Using Euclidean algorithm and theory of linear Diophantine equation, divide 100 into two summands such that one is divisible by 7 and other by 11.

(c) Write a short note on Prime number theorem.

(d) If $ca \equiv cb \pmod{n}$ then prove that $a \equiv b \pmod{n/d}$, where $d = \gcd(c, n)$.

2. (a) Verify that $0, 1, 2, 2^2, 2^3 \ldots\ldots\ldots 2^9$ form a complete set of residues modulo 11, but that $0, 1^2, 2^2, 3^2 \ldots\ldots 10^2$ do not.

(b) Find the solutions of the system of congruences :

$$3x + 4y \equiv 5 \pmod{13}$$

$$2x + 5y \equiv 7 \pmod{13}.$$

(c) Use Fermat's theorem to verify that 17 divides $11^{104} + 1$.

(d) Find the remainder when $2(26!)$ is divided by 29.

3. (a) Let F and f be two number – theoretic functions related by the formula

$$F(n) = \sum_{d/n} f(d)$$

Prove $f(n) = \Sigma_{d|n}\mu(d)F(n/d) = \Sigma_{d|n}\mu(n/d)F(d)$.

(b) Verify that 1000! terminates in 249 zeros.

(c) Use Euler's theorem for any integer a, to prove that $a^{13} \equiv a \pmod{2730}$

(d) Prove that $\emptyset(2^n - 1)$ is a multiple of n for any $n > 1$.

4. (a) For any positive integer n, prove

$\emptyset(n) = n\Sigma_{d|n}\mu(d)/d$.

(b) Define primitive roots of an integer by an example and show that if $F_n = 2^{2^n} + 1$, $n > 1$, is a prime then 2 is not a primitive root of F_n.

(c) If p is a prime number and $d|p-1$, then show that there are exactly $\varphi(d)$ incongruent integers having order d modulo p.

(d) Determine all the primitive roots of the primes $p = 11, 19$, and 23, expressing each as a power of one of the roots.

5. (a) If gcd (m, n) = 1, where $m > 2$ and $n > 2$, then prove that the integer 'mn' has no primitive roots.

(b) Solve the quadratic congruence

$3x^2 + 9x + 7 \equiv 0 \pmod{13}$.

(c) Show that 3 is quadratic residue of 23, but a nonresidue of 31.

(d) Prove that there are infinitely many primes of the form 4k+1.

6. (a) Find the value of Legendre symbol (1234/4567).

(b) Solve the quadratic congruence

$x^2 \equiv 23 \pmod{7^3}$.

(c) Using the linear cipher $C \equiv 5P + 11 \pmod{26}$, encrypt the message NUMBER THEORY IS EASY.

(d) When the RSA algorithm is based on the key $(n, k) = (3233, 37)$, what is the recovery exponent for the cryptosystem?